Math 249 Lecture 30 Notes

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1 The Plethystic Logarithm

To finish up our discussion on the theory of plethystic substitution for now, we need to briefly discuss the Möbius function.

1.1 The Möbius function

Definition 1.1. The *Möbius funcction* $\mu(k)$ is defined recursively by

$$\sum_{k|n} \mu(k) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1. \end{cases}$$

Example 1.1. Let's compute some values of μ:

n	1	2	3	4	5	6	7	8	9	$\cdots$
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	$\cdots$

Proposition 1.1.

$$\mu(n) = \begin{cases} (-1)^{\text{number of prime factors of } n} & n \text{ is square-free} \\ 0 & \text{otherwise} \end{cases}$$

We will not prove this, but you can do the proof yourself for fun. [footnote about mobius inversion]

1.2 The plethystic logarithm

Last time we said that we can find the cycle index for connected graphs by manipulating the fact that a graph is a union of connected graphs:

$$Z_G = \underbrace{Z_E}_{\Omega} * Z_{G_c}.$$

In other words, we want to solve the equation

$$B = \Omega * A.$$

Proposition 1.2. *If we have the equation $B = \Omega[A]$,*

$$A = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log p_k[\Omega[A]]$$

Proof. Note that

$$\Omega[A] = \exp \sum_{\ell=1}^{\infty} \frac{1}{\ell} p_\ell[A],$$

$$p_k[\Omega[A]] = \exp \sum_{\ell=1}^{\infty} \frac{1}{\ell} p_{k\ell}[A].$$

So the right hand side is

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \sum_{\ell=1}^{\infty} \frac{1}{\ell} p_{k\ell}[A] &= \sum_{k\ell=1}^{\infty} \frac{\mu(k)}{k\ell} p_{k\ell}[A] \\ &= \sum_{n=1}^{\infty} \sum_{k|n} \mu(k) \frac{p_n[A]}{n} \\ &= p_1[A] \\ &= A. \end{aligned}$$

□

If we call the right hand side Λ, then what this says is that

$$B = \Omega[A] \implies A = \Lambda[B].$$

2 Coxeter groups

2.1 Definition and examples

These are also called *finite real reflection groups.*

Definition 2.1. Given a nonzero $v \in \mathbb{R}^n$, define the hyperplane $H_v = \{w : (w, v) = 0\}$. A *reflection* s_H across H is a transformation such that

$$s_H(w) = \begin{cases} w & w \in H_v \\ -v & w = v. \end{cases}$$

Definition 2.2. A *Coxeter group* G is a subgroup of $O_n(\mathbb{R})$ that is generated by reflections.

Coexeter groups come with a faithful representation $G \circlearrowright \mathbb{R}^n$.

Example 2.1. S_n is a Coxeter group. It acts on $\mathbb{R}^n$ by permuting the basis vectors. So for the transposition $(i\ j) \in S_n$ and basis $\{e_1, \ldots, e_n\} \subseteq \mathbb{R}^n$, we have

$$e_k \mapsto e_k \quad \text{if } k \neq i, j$$

$$e_i + e_j \mapsto e_i + e_j,$$
$$e_i - e_j \mapsto -(e_i - e_j).$$

The corresponding vector and orthogonal hyperplane are

$$v = e_i - e_j$$

$$H = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = x_j\}.$$

The vector $(1, 1, \ldots, 1)$ is invariant, so

$$H = \mathbb{R} \cdot (1, 1, \ldots, 1)^{\perp} = \{(x_1, \ldots, x_n) : x_1 + \cdots + x_n = 0\} \cong \mathbb{R}^{n-1}.$$

In fact, $(i\ j)$ is still a reflection on this hyperplane.

Example 2.2. Let $S_3 \circlearrowright \mathbb{R}^3$ with basis vectors v_1, v_2, v_3. We can draw 3 perpendicular planes. If you draw a triangle on these (with vertices at the tip of a unit vector on each plane), you can see that S_3, which permutes the planes, is the set of symmetries of the triangle; i.e. $S_3 \cong D_6$.

Example 2.3. The dihedral groups D_{2k} are Coexeter groups. They are generated by k reflections, and the also have k rotations (which are each the product of 2 reflections).

If $G \circlearrowright \mathbb{R}^m$ nad $H \circlearrowright \mathbb{R}^n$, then $G \times H \circlearrowright \mathbb{R}^m \oplus \mathbb{R}^n$. So the product of Coxeter groups is a Coxeter group.

Example 2.4. The signed permutations $B_n \circlearrowright \mathbb{R}^n$ are a permutation group. They are generated by S_n and matrices of the form

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

In fact, $B_n \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. This is non-degenerate; no vector is fixed under this action.